

Dirac-Kähler particle in Riemann spherical space: boson interpretation

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November 26, 2014

Abstract

In the context of the composite boson interpretation, we construct the exact general solution of the Dirac-Kähler equation for the case of the spherical Riemann space of constant positive curvature, for which due to the geometry itself one may expect to have a discrete energy spectrum. In the case of the minimum value of the total angular momentum, $j = 0$, the radial equations are reduced to second-order ordinary differential equations, which are straightforwardly solved in terms of the hypergeometric functions. For non-zero values of the total angular momentum, however, the radial equations are reduced to a pair of complicated fourth-order differential equations. Employing the factorization approach, we derive the general solution of these equations involving four independent fundamental solutions written in terms of combinations of the hypergeometric functions. The corresponding discrete energy spectrum is then determined via termination of the involved hypergeometric series, resulting in quasi-polynomial wave-functions. The constructed solutions lead to notable observations when compared with those for the ordinary Dirac particle. The energy spectrum for the Dirac-Kähler particle in spherical space is much more complicated. Its structure substantially differs from that for the Dirac particle since it consists of two paralleled energy level series each of which is twofold degenerate. Besides, none of the two separate series coincides with the series for the Dirac particle. Thus, the Dirac-Kähler field cannot be interpreted as a system of four Dirac fermions. Additional arguments supporting this conclusion are discussed.

PACS numbers: 04.62.+v Quantum fields in curved spacetime, 02.40.Ky Riemannian geometries, 03.65.Ge Solutions of wave equations: bound states

Keywords: fermion, boson, Dirac-Kähler particle, spherical Riemann space, exact solution, factorization method, hypergeometric function, energy spectrum

1 Introduction

The concept of elementary particles as certain relativistically invariant objects in the frame of the 4-dimensional Minkowski space-time is well appreciated already for a long time. For any particle, it assumes certain transformation properties of a corresponding field, and a wave equation that describes this field.

An interesting problem here, that attracted much attention in past, is the extension of the wave equation for the Dirac-Kähler field. The literature devoted to this field is huge (see, e.g., [1–33]). The discussion started already with the early development of the theory of quantum mechanical wave equations, just after the concept of a particle with spin $1/2$ appeared [1]. The Dirac-Kähler field was invented yet in 1928–1929 by Darwin [2], Ivanenko and Landau [3] as an alternative to the Dirac field. The main intention was to construct a wave equation for a spin $1/2$ particle on

the basis of tensor objects without using spinors. In particular, an argument was that the new objects, the spinors, seemed mysterious and obscure in comparison to the familiar tensors. The Dirac–Kähler field consists of a set of tensors, which is equivalent to a 2-rank bispinor. It contains 16 independent components, and its wave equation can be presented formally as four not-connected Dirac-type equations. The main feature of the Dirac–Kähler field is that it gives a possibility to perform (seemingly) smooth transition from tensors to spinors, however, in a sense, it was an attempt to eliminate spinors at all. This was the main ground to consider this object as related to fermions. However, the mentioned disconnection of the four involved Dirac-type equations is valid only for the case of flat Minkowski space-time, and this feature is not preserved in the presence of a gravitational field, that is for any non-Euclidean space-time model.

The three most interesting points in connection with the general covariant extension of the wave equation for this field are: first, in flat Minkowski space there exist tensor and spinor formulations of the theory; second, in the initial tensor form there are tensors with different intrinsic parities; and third, there exist different views about physical interpretation of the object: whether it is a composite boson or a set of four fermions.

In the Minkowski space, the Dirac–Kähler particle is described by 16-component wave function $U(x)$, a bispinor of rank 2, or by equivalent set of tensor fields: $\{\Phi(x), \Phi_i(x), \tilde{\Phi}(x), \tilde{\Phi}_i(x), \Phi_{mn}(x)\}$, where $\Phi(x)$ is a scalar, $\Phi_i(x)$ is a vector, $\tilde{\Phi}(x)$ represents a pseudo-scalar, $\tilde{\Phi}_i(x)$ represents a pseudo-vector, $\Phi_{mn}(x)$ is an antisymmetric tensor. Connection between these quantities is given by the formula [33]

$$U = \left(-i\Phi + \gamma^l \Phi_l + i\sigma^{mn} \Phi_{mn} + \gamma^5 \tilde{\Phi} + i \gamma^l \gamma^5 \tilde{\Phi}_l \right) E^{-1}, \quad (1.1)$$

where $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, $\sigma^{ab} = \frac{1}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$, and E stands for bispinor metrical matrix [33].

For a curved space-time background, the generally covariant tetrad-based Dirac–Kähler equation in 4-spinor form has the form [33]

$$[i\gamma^\alpha(x) (\partial/\partial x^\alpha + B_\alpha(x)) - m] U(x) = 0, \quad (1.2)$$

where B_α is the 2-bispinor connection: $B_\alpha = \frac{1}{2}J^{ab}e_{(a)}^\beta \nabla_\alpha(e_{(b)\beta})$ with $J^{ab} = \sigma^{ab} \otimes I + I \otimes \sigma^{ab}$ standing for generators of second rank bispinor under the Lorentz group. This spinor equation is equivalent to the following generally covariant tensor system [33]:

$$\begin{aligned} \nabla^\alpha \Psi_\alpha + m\Psi &= 0, & \nabla^\alpha \tilde{\Psi}_\alpha + m\tilde{\Psi} &= 0, \\ \nabla_\alpha \Psi + \nabla^\beta \Psi_{\alpha\beta} - m\Psi_\alpha &= 0, \\ \nabla_\alpha \tilde{\Psi} - \frac{1}{2}\epsilon_\alpha^{\beta\rho\sigma}(x)\nabla_\beta \Psi_{\rho\sigma} - m\tilde{\Psi}_\alpha &= 0, \\ \nabla_\alpha \Psi_\beta - \nabla_\beta \Psi_\alpha + \epsilon_{\alpha\beta}^{\rho\sigma}(x)\nabla_\rho \tilde{\Psi}_\sigma - m\Psi_{\alpha\beta} &= 0. \end{aligned} \quad (1.3)$$

The covariant tensor field variables are connected with the local tetrad tensor variables by the relations

$$\Psi_\alpha = e_\alpha^{(i)} \Psi_i, \quad \tilde{\Psi}_\alpha = e_\alpha^{(i)} \tilde{\Psi}_i, \quad \Psi_{\alpha\beta} = e_\alpha^{(m)} e_\beta^{(n)} \Psi_{mn}, \quad (1.4)$$

and the Levi-Civita object is determined as $\epsilon^{\alpha\beta\rho\sigma}(x) = \epsilon^{abcd} e_{(a)}^\alpha e_{(b)}^\beta e_{(c)}^\rho e_{(d)}^\sigma$. The fields Ψ , Ψ_α , $\Psi_{\alpha\beta}$ are tetrad scalars, $\tilde{\Psi}$, $\tilde{\Psi}_\alpha$ are tetrad pseudo-scalars, the Levi-Civita object $\epsilon^{\alpha\beta\rho\sigma}(x)$ is simultaneously a generally covariant tensor and a tetrad pseudo-scalar.

The most of the earlier work on this object concentrates on the symmetries and other fundamental aspects of the relationships between the Dirac and Dirac-Kähler fields [1–33], however, there are no known non-trivial treatments for the Dirac-Kähler equation when discussing the Dirac-Kähler field on curved space-time backgrounds. However, the curved space-time models are of primary importance, because on a non-Euclidean geometric background only a composite boson interpretation for the Dirac-Kähler field is possible.

In the present paper we construct an exact solution for the case of the simplest non-Euclidean geometrical model - the spherical Riemann space of positive curvature. Due to the geometry itself, one may expect to get discrete energy spectrum for a Dirac-Kähler particle. The latter supposition turns out to be the case. Below we construct, in terms of the hypergeometric functions, the general solution of the problem consisting of four particular fundamental solutions, each of which allows a separate discrete energy level series. The determined spectrum for a Dirac-Kähler particle reveals substantial peculiarities as compared with the spectrum for the ordinary Dirac case. It exhibits a double-series structure, each of the series being twofold degenerate, and, moreover, none of the energy level series coincides with that one for the Dirac particle.

2 Separation of the variables

The known results concerning the separation of the variables for the Riemann spherical space are as follows. The Dirac-Kähler equation in the Minkowski space is written as [33]

$$\left[i\gamma^0 \partial_t + i \left(\gamma^3 \partial_r + \frac{\gamma^1 J^{31} + \gamma^2 J^{32}}{r} \right) + \frac{1}{r} \Sigma_{\theta,\phi} - m \right] U(x) = 0 , \quad (2.1)$$

$$\Sigma_{\theta,\phi} = i\gamma^1 \partial_\theta + \gamma^2 \frac{i\partial_\phi + iJ^{12} \cos \theta}{\sin \theta} , \quad J^{12} = (\sigma^{12} \otimes I + I \otimes \sigma^{12}) . \quad (2.2)$$

Diagonalizing the operators of the total angular momentum

$$J_1 = l_1 + \frac{iJ^{12} \cos \phi}{\sin \theta} , \quad J_2 = l_2 + \frac{iJ^{12} \sin \phi}{\sin \theta} , \quad J_3 = l_3 , \quad (2.3)$$

for wave function we have the following general representation:

$$U_{\epsilon jm}(t, r, \theta, \phi) = \frac{e^{-i\epsilon t}}{r} \begin{vmatrix} f_{11} D_{-1} & f_{12} D_0 & f_{13} D_{-1} & f_{14} D_0 \\ f_{21} D_0 & f_{22} D_{+1} & f_{23} D_0 & f_{24} D_{+1} \\ f_{31} D_{-1} & f_{32} D_0 & f_{33} D_{-1} & f_{34} D_0 \\ f_{41} D_0 & f_{42} D_{+1} & f_{43} D_0 & f_{44} D_{+1} \end{vmatrix} , \quad (2.4)$$

where $f_{ab} = f_{ab}(r)$, $D_\sigma = D_{-m,\sigma}^j(\phi, \theta, 0)$ stands for the Wigner functions [35], and j takes the values 0, 1, 2, The next step is to diagonalize the space reflection operator. In the spherical tetrad basis it has the form

$$\hat{\Pi} = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} \otimes \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} \otimes \hat{P} . \quad (2.5)$$

The eigenvalue equation $\hat{\Pi} U_{\epsilon jm} = \Pi U_{\epsilon jm}$ imposes the following restrictions:

$$\begin{aligned} f_{31} &= \pm f_{24} , & f_{32} &= \pm f_{23} , & f_{33} &= \pm f_{22} , & f_{34} &= \pm f_{21} , \\ f_{41} &= \pm f_{14} , & f_{42} &= \pm f_{13} , & f_{43} &= \pm f_{12} , & f_{44} &= \pm f_{11} , \end{aligned} \quad (2.6)$$

where the upper sign refers to the eigenvalue $\Pi = (-1)^{j+1}$, and the lower sign refers to $\Pi = (-1)^j$. Eq. (2.4) is rewritten as

$$U_{\epsilon jm\delta}(t, r, \theta, \phi) = \frac{e^{-i\epsilon t}}{r} \begin{vmatrix} f_{11} D_{-1} & f_{12} D_0 & f_{13} D_{-1} & f_{14} D_0 \\ f_{21} D_0 & f_{22} D_{+1} & f_{23} D_0 & f_{24} D_{+1} \\ \delta f_{24} D_{-1} & \delta f_{23} D_0 & \delta f_{22} D_{-1} & \delta f_{21} D_0 \\ \delta f_{14} D_0 & \delta f_{13} D_{+1} & \delta f_{12} D_0 & \delta f_{11} D_{+1} \end{vmatrix}, \quad (2.7)$$

where $\delta = +1$ refers to the case $\Pi = (-1)^j$ and $\delta = -1$ stands for the case $\Pi = (-1)^{j+1}$.

The system of radial equations for $\delta = +1$ reads

$$\begin{aligned} \epsilon f_{24} - i \frac{d}{dr} f_{24} + \frac{i}{r} 0 - \frac{ia}{r} f_{14} - m f_{11} &= 0, & \epsilon f_{23} - i \frac{d}{dr} f_{23} - \frac{i}{r} f_{14} - \frac{ia}{r} f_{13} - m f_{12} &= 0, \\ \epsilon f_{14} + i \frac{d}{dr} f_{14} + \frac{i}{r} f_{23} + \frac{ia}{r} f_{24} - m f_{21} &= 0, & \epsilon f_{13} + i \frac{d}{dr} f_{13} + \frac{i}{r} 0 + \frac{ia}{r} f_{23} - m f_{22} &= 0, \\ \epsilon f_{22} - i \frac{d}{dr} f_{22} + \frac{i}{r} 0 - \frac{ia}{r} f_{12} - m f_{13} &= 0, & \epsilon f_{21} - i \frac{d}{dr} f_{21} - \frac{i}{r} f_{12} - \frac{ia}{r} f_{11} - m f_{14} &= 0, \\ \epsilon f_{12} + i \frac{d}{dr} f_{12} + \frac{i}{r} f_{21} + \frac{ia}{r} f_{22} - m f_{23} &= 0, & \epsilon f_{11} + i \frac{d}{dr} f_{11} + \frac{i}{r} 0 + \frac{ia}{r} f_{21} - m f_{24} &= 0. \end{aligned} \quad (2.8)$$

The set of the equations for the case $\delta = -1$ is readily obtained from this system by the formal change $m \rightarrow -m$.

Introducing the notations

$$\begin{aligned} A &= (f_{11} + f_{22}), & iB &= (f_{11} - f_{22}), & C &= (f_{12} + f_{21}), & iD &= (f_{12} - f_{21}), \\ K &= (f_{13} + f_{24}), & iL &= (f_{13} - f_{24}), & M &= (f_{14} + f_{23}), & iN &= (f_{14} - f_{23}), \end{aligned} \quad (2.9)$$

the system (2.8) is changed to a form without complex coefficients:

$$\begin{aligned} \epsilon K - \frac{dL}{dr} + \frac{a}{r} N - mA &= 0, & \epsilon A - \frac{dB}{dr} + \frac{a}{r} D - mK &= 0, \\ \epsilon L + \frac{dK}{dr} + \frac{a}{r} M + mB &= 0, & \epsilon B + \frac{dA}{dr} + \frac{a}{r} C + mL &= 0, \\ \epsilon M - \frac{dN}{dr} + \frac{1}{r} N + \frac{a}{r} L - mC &= 0, & \epsilon C - \frac{dD}{dr} + \frac{1}{r} D + \frac{a}{r} B - mM &= 0, \\ \epsilon N + \frac{dM}{dr} + \frac{1}{r} M + \frac{a}{r} K + mD &= 0, & \epsilon D + \frac{dC}{dr} + \frac{1}{r} C + \frac{a}{r} A + mN &= 0. \end{aligned} \quad (2.10)$$

These equations permit imposing of the following linear constraints:

$$A = \lambda K, \quad B = \lambda L, \quad C = \lambda M, \quad D = \lambda N \quad (2.11)$$

with $\lambda = \pm 1$. For the case $\lambda = +1$, instead of (2.10) we then derive four equations:

$$\begin{aligned} \frac{dK}{dr} + \frac{a}{r} M + (\epsilon + m)L &= 0, & \frac{dL}{dr} - \frac{a}{r} N - (\epsilon - m)K &= 0, \\ \left(\frac{d}{dr} + \frac{1}{r} \right) M + \frac{a}{r} K + (\epsilon + m)N &= 0, & \left(\frac{d}{dr} - \frac{1}{r} \right) N - \frac{a}{r} L - (\epsilon - m)M &= 0. \end{aligned} \quad (2.12)$$

If here m is changed to $-m$, one gets a corresponding set of four equations for the case $\lambda = -1$.

Eqs. (2.12) can be further reduced as follows:

$$\begin{aligned}
(i) \quad & \sqrt{j+1} K(r) = f(r) \ , \ \sqrt{j+1} L(r) = g(r) \ , \\
& \sqrt{j} M(r) = f(r) \ , \ \sqrt{j} N(r) = g(r) \ , \\
& \left(\frac{d}{dr} + \frac{j+1}{r} \right) f + (\epsilon + m)g = 0 \ , \\
& \left(\frac{d}{dr} - \frac{j+1}{r} \right) g - (\epsilon - m)f = 0 \ ;
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
(ii) \quad & \sqrt{j} K(r) = f(r) \ , \ \sqrt{j} L(r) = g(r) \ , \\
& \sqrt{j+1} M(r) = -f(r) \ , \ \sqrt{j+1} N(r) = -g(r) \ , \\
& \left(\frac{d}{dr} - \frac{j}{r} \right) f + (\epsilon + m)g = 0 \ , \\
& \left(\frac{d}{dr} + \frac{j}{r} \right) g - (\epsilon - m)f = 0 \ .
\end{aligned} \tag{2.14}$$

The derived equations apply to the case of nonzero j . The case $j = 0$ needs a separate treatment because the Wigner functions $D_{0,\pm 1}^j$ with $j = 0$ are not well posed. The initial substitution in this case is simpler:

$$U_{\epsilon 00}(t, r) = \frac{e^{-i\epsilon t}}{r} \begin{vmatrix} 0 & f_{12} & 0 & f_{14} \\ f_{21} & 0 & f_{23} & 0 \\ 0 & f_{32} & 0 & f_{34} \\ f_{41} & 0 & f_{43} & 0 \end{vmatrix} . \tag{2.15}$$

Space reflection operator divides the solutions (2.15) into two classes:

$$\Pi = +1 \ (\delta = +1), \quad f_{32} = +f_{23} \ , \ f_{34} = +f_{21} \ , \ f_{41} = +f_{14} \ , \ f_{43} = +f_{12} \ ; \tag{2.16}$$

$$\Pi = -1 \ (\delta = -1), \quad f_{32} = -f_{23} \ , \ f_{34} = -f_{21} \ , \ f_{41} = -f_{14} \ , \ f_{43} = -f_{12} \ . \tag{2.17}$$

Using the identity $\Sigma_{\theta, \phi} U_{\epsilon 00} = 0$, we get the radial system

$$\begin{aligned}
\epsilon M - \frac{dN}{dr} + \frac{N}{r} - m C &= 0 \ , \ \epsilon N + \frac{dM}{dr} + \frac{M}{r} + m D = 0 \ , \\
\epsilon C - \frac{dD}{dr} + \frac{D}{r} - m M &= 0 \ , \ \epsilon D + \frac{dC}{dr} + \frac{C}{r} - m N = 0 \ .
\end{aligned} \tag{2.18}$$

This system is further reduced as follows:

$$C = +M \ , \ D = +N \ (\lambda = +1)$$

$$\left(\frac{d}{dr} + \frac{1}{r} \right) M + (\epsilon + m)N = 0 \ , \quad \left(\frac{d}{dr} - \frac{1}{r} \right) N - (\epsilon - m)M = 0 \ ; \tag{2.19}$$

$$C = -M \ , \ D = -N \ (\lambda = -1)$$

$$\left(\frac{d}{dr} + \frac{1}{r} \right) M + (\epsilon - m)N = 0 \ , \quad \left(\frac{d}{dr} - \frac{1}{r} \right) N - (\epsilon + m)M = 0 \ . \tag{2.20}$$

Thus, we have established the disconnection of the radial equations. However, we stress that this is only possible in the Minkowski space.

The above development is readily extended to the case of the spherical Riemann space model. The metric is given as

$$dS^2 = dt^2 - dr^2 - \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2) , \quad (2.21)$$

and the Dirac-Kähler equation for this geometry is written as

$$\left[i\gamma^0 \partial_t + i \left(\gamma^3 \partial_r + \frac{\gamma^1 J^{31} + \gamma^2 J^{32}}{\tan r} \right) + \frac{1}{\sin r} \Sigma_{\theta, \phi} - m \right] U(x) = 0 . \quad (2.22)$$

The quantum number of the total angular momentum adopts the values $j = 0, 1, 2, \dots$.

The calculations are much the same as above (see the detailed derivation in [33]), and differences appear only in the explicit form of the final systems of radial equations.

For the case of minimum $j = 0$, the problem is reduced to the system

$$\begin{aligned} \epsilon M - \frac{dN}{dr} + \frac{N}{\tan r} - m C &= 0 , \quad \epsilon N + \frac{dM}{dr} + \frac{M}{\tan r} + m D = 0 , \\ \epsilon C - \frac{dD}{dr} + \frac{D}{\tan r} - m M &= 0 , \quad \epsilon D + \frac{dC}{dr} + \frac{C}{\tan r} - m N = 0 , \end{aligned} \quad (2.23)$$

while for greater total angular momenta, i.e. for $j = 1, 2, 3, \dots$, the problem reduces to the following more complicated equations ($a = \sqrt{j(j+1)}$):

$$\begin{aligned} \frac{dK}{dr} + \frac{a}{\sin r} M + (\epsilon + m)L &= 0 , \\ \frac{dL}{dr} - \frac{a}{\sin r} N - (\epsilon - m)K &= 0 , \\ \left(\frac{d}{dr} + \frac{1}{\tan r} \right) M + \frac{a}{\sin r} K + (\epsilon + m)N &= 0 , \\ \left(\frac{d}{dr} - \frac{1}{\tan r} \right) N - \frac{a}{\sin r} L - (\epsilon - m)M &= 0 . \end{aligned} \quad (2.24)$$

Note that the above analysis of the spherical model is also applicable if an external spherically symmetric potential is added. In this case the only needed formal change is $\epsilon \rightarrow \epsilon - U(r)$.

3 The case of minimum total angular momentum $j = 0$

If $j = 0$, the system (2.23) is reduced using two substitutions:

$$C = +M , \quad D = +N \quad (\lambda = +1),$$

$$\left(\frac{d}{dr} + \frac{1}{\tan r} \right) M + (\epsilon + m)N = 0 , \quad \left(\frac{d}{dr} - \frac{1}{\tan r} \right) N - (\epsilon - m)M = 0 , \quad (3.1)$$

$$C = -M , \quad D = -N \quad (\lambda = -1),$$

$$\left(\frac{d}{dr} + \frac{1}{\tan r} \right) M + (\epsilon - m)N = 0 , \quad \left(\frac{d}{dr} - \frac{1}{\tan r} \right) N - (\epsilon + m)M = 0 . \quad (3.2)$$

The two first-order equations of system (3.1) lead to the following second-order equations for M and N :

$$\left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 - \frac{1 + \cos^2 r}{\sin^2 r} \right) M = 0, \quad (3.3)$$

$$\left(\frac{d^2}{dr^2} + \epsilon^2 - m^2 + 1 \right) N = 0, \quad N = \text{const } e^{\pm i\sqrt{\epsilon^2 - m^2 + 1} r}. \quad (3.4)$$

Eq.(3.3) is solved in terms of the hypergeometric functions. Indeed, passing to a new independent variable $x = (1 - \cos r)/2$:

$$x(1-x) \frac{d^2 M}{dx^2} + \left(\frac{1}{2} - x \right) \frac{dM}{dx} + \left(\epsilon^2 - m^2 + 1 - \frac{1}{2x} - \frac{1}{2(1-x)} \right) M = 0, \quad (3.5)$$

and applying the substitution $M(x) = x^A(1-x)^B F(x)$ with $A = -\frac{1}{2}$, 1 , $B = -\frac{1}{2}$, 1 , we arrive at the Gauss hypergeometric equation for $F(x) = F(\alpha, \beta; \gamma; x)$ with parameters

$$\alpha = 2 - \sqrt{\epsilon^2 - m^2 + 1}, \quad \beta = 2 + \sqrt{\epsilon^2 - m^2 + 1}, \quad \gamma = \frac{5}{2}. \quad (3.6)$$

In order to have a solution which is finite on the spherical space with $r \in [0, \pi]$, one should take only the positive values $A = 1$ and $B = 1$. The condition for the solutions to turn into polynomials, $\alpha = -n$, $n = 0, 1, 2, \dots$, gives the energy spectrum for the states with $j = 0$:

$$\epsilon^2 - m^2 + 1 = (2 + n)^2, \quad n = 0, 1, 2, 3, \dots \quad (3.7)$$

To accomplish the development, one should determine the coefficient connecting the functions $M(r)$ and $N(r)$. With straightforward calculations, we get the following result:

$$N = N_0 \sqrt{x(1-x)} F(-n-1, 3+n; 3/2; x), \\ M = M_0 x(1-x) F(-n, 4+n; 5/2; x), \quad M_0/N_0 = -(2/3)(\epsilon - m). \quad (3.8)$$

4 Analysis of the radial equations for $j = 1, 2, \dots$

With the help of the first two equations of the system (2.24) we eliminate the functions L and N :

$$L = -\frac{1}{\epsilon + m} \left(\frac{d}{dx} K + \frac{a}{\sin x} M \right), \\ N = -\frac{1}{\epsilon + m} \left(\frac{d}{dx} M + \frac{1}{\tan x} M + \frac{a}{\sin x} K \right), \quad (4.1)$$

and arrive at two equations relating K and M (we use the notation $\epsilon^2 - m^2 = p^2$, $p > 0$):

$$\left(\frac{d^2}{dx^2} + p^2 - \frac{a^2}{\sin^2 x} \right) K = \frac{2a \cos x}{\sin^2 x} M, \\ \left(\frac{d^2}{dx^2} + p^2 + 1 - \frac{a^2 + 2}{\sin^2 x} \right) M = \frac{2a \cos x}{\sin^2 x} K. \quad (4.2)$$

Rewriting Eqs.(4.2) for the variable $x = \cos^2 r$, we obtain

$$\left((1-x)4x \frac{d^2}{dx^2} + 2(1-2x) \frac{d}{dx} + p^2 - \frac{a^2}{1-x} \right) K = \frac{2a\sqrt{x}}{1-x} M, \quad (4.3)$$

$$\left((1-x)4x \frac{d^2}{dx^2} + 2(1-2x) \frac{d}{dx} + p^2 + 1 - \frac{a^2+2}{1-x} \right) M = \frac{2a\sqrt{x}}{1-x} K. \quad (4.4)$$

Near the point $x = 1$, the asymptotic behavior of the solutions is given as

$$M = M_0(1-x)^\gamma, \quad K = K_0(1-x)^\gamma.$$

Eqs. (4.3) and (4.4) then lead to

$$(4\gamma^2 - 2\gamma - a^2)K_0 - 2aM_0 = 0, \quad -2aK_0 + (4\gamma^2 - 2\gamma - a^2 - 2)M_0 = 0. \quad (4.5)$$

Whence, we get four possible values for γ : $\gamma = j/2$, $(j+2)/2$, $(-j+1)/2$, $(-j-1)/2$. Only the positive values $\gamma = j/2$, $(j+2)/2$ may describe bound states.

Elimination of $M(x)$ from Eqs.(4.3)-(4.4) results in the following fourth-order ordinary differential equation for $K(x)$:

$$\begin{aligned} x^2 \frac{d^4 K}{dx^4} + \left(7x + 5 - \frac{5}{1-x} \right) \frac{d^3 K}{dx^3} + \left(10 - \frac{1}{2} p^2 + \frac{p^2 + a^2 - 28}{2(1-x)} + \frac{15 - 2a^2}{4(1-x)^2} \right) \frac{d^2 K}{dx^2} + \\ + \left(\frac{1}{4x} + \frac{3p^2 - 7}{4(1-x)} - \frac{3p^2 + a^2 - 9}{4(1-x)^2} + \frac{a^2}{4(1-x)^3} \right) \frac{dK}{dx} + \\ + \left(\frac{p^2 - a^2}{8x} + \frac{p^2 - a^2}{8(1-x)} + \frac{p^4 + 2p^2 - 2a^2}{16(1-x)^2} - \frac{a^2(p^2 - 1)}{8(1-x)^3} + \frac{a^2(a^2 - 2)}{16(1-x)^4} \right) K = 0, \end{aligned} \quad (4.6)$$

In the similar manner, elimination of $K(x)$ gives the following equation for $M(x)$:

$$\begin{aligned} x^2 \frac{d^4 M}{dx^4} + \left(7x + 5 - \frac{5}{1-x} \right) \frac{d^3 M}{dx^3} + \left(10 - \frac{1}{2} p^2 + \frac{p^2 + a^2 - 28}{2(1-x)} + \frac{15 - 2a^2}{4(1-x)^2} \right) \frac{d^2 M}{dx^2} + \\ + \left(\frac{1}{4x} + \frac{3p^2 - 6}{4(1-x)} - \frac{3p^2 + a^2 - 9}{4(1-x)^2} + \frac{a^2}{4(1-x)^3} \right) \frac{dM}{dx} + \\ + \left(\frac{p^2 - a^2 - 1}{8x} + \frac{p^2 - a^2 - 1}{8(1-x)} + \frac{p^4 + 2p^2 - 2a^2 - 3}{16(1-x)^2} - \frac{a^2(p^2 - 1)}{8(1-x)^3} + \frac{a^2(a^2 - 2)}{16(1-x)^4} \right) M = 0. \end{aligned} \quad (4.7)$$

5 General solution via factorization

A tool for studying such complicated equations is the operator factorization method when one tries to present the higher-order differential operators as products of lower-order ones. In our case this leads to the general solution of the problem if one tries the following factorization:

$$\begin{aligned} \hat{K}_4 K(x) = \hat{A}_2 \hat{K}_2 K(x) = \hat{A}_2 f(x) = 0, \\ \hat{M}_4 M(x) = \hat{B}_2 \hat{M}_2 M(x) = \hat{B}_2 g(x) = 0. \end{aligned} \quad (5.1)$$

Obviously, the solutions of the second-order equations

$$f(x) = \hat{K}_2 K(x) = 0 \quad \text{and} \quad g(x) = \hat{M}_2 K(x) = 0$$

resolve, respectively, the equations $\hat{K}_4 K(x) = 0$ and $\hat{M}_4 M(x) = 0$. Though generally one may expect to derive in this way only a part of all solutions, we will see that in the present case this approach allows complete solution of the problem.

It is readily checked that Eq.(4.6) for $K(x)$ is factorized as follows:

$$\left[\frac{d^2}{dx^2} + \frac{1}{2} \left(\frac{3}{x} - \frac{7}{1-x} \right) \frac{d}{dx} + \frac{1}{4} \left(\frac{p^2 - a^2 - 10}{x} + \frac{p^2 - a^2 - 10}{1-x} - \frac{a^2 - 6}{(1-x)^2} \right) \right] f(x) = 0, \quad (5.2)$$

where

$$f(x) \equiv \left[\frac{d^2}{dx^2} + \frac{1}{2} \left(\frac{1}{x} - \frac{3}{1-x} \right) \frac{d}{dx} + \frac{1}{4} \left(\frac{p^2 - a^2}{x} + \frac{p^2 - a^2}{1-x} - \frac{a^2}{(1-x)^2} \right) \right] K(x). \quad (5.3)$$

Further, the equation $f(x) = 0$ is solved in terms of the hypergeometric functions. To derive this solution, we make the substitution $K(x) = x^A(1-x)^B F(x)$ with

$$A = 0, \quad \frac{1}{2}, \quad B = -\frac{1}{4} \pm \frac{1}{4} \sqrt{4a^2 + 1}, \quad F(x) = F(\alpha, \beta; \gamma; x), \quad (5.4)$$

$$\alpha = A + B + \frac{1}{2} - \frac{1}{2} \sqrt{p^2 + 1}, \quad \beta = A + B + \frac{1}{2} + \frac{1}{2} \sqrt{p^2 + 1}, \quad \gamma = \frac{1}{2} + 2A. \quad (5.5)$$

Each pair (A, B) of possible four such choices produces a solution of Eq.(4.6). However, we are interested in finite and continuous solutions at $x = 0$ and $x = 1$, therefore, we take the following two pairs which produce two independent solutions of Eq.(4.6), $K_1(x)$ and $K_2(x)$:

$$(i) \quad A = \frac{1}{2}, \quad B = -\frac{1}{4} + \frac{1}{4} \sqrt{4a^2 + 1} = +\frac{j}{2} > 0,$$

$$K_1 = x^{1/2} (1-x)^{j/2} F \left(1 + j/2 - \frac{1}{2} \sqrt{p^2 + 1}, 1 + j/2 + \frac{1}{2} \sqrt{p^2 + 1}, 3/2, x \right); \quad (5.6)$$

$$(ii) \quad A = 0, \quad B = -\frac{1}{4} + \frac{1}{4} \sqrt{4a^2 + 1} = +\frac{j}{2} > 0,$$

$$K_2 = (1-x)^{j/2} F \left(j/2 + 1/2 - \frac{1}{2} \sqrt{p^2 + 1}, j/2 + 1/2 + \frac{1}{2} \sqrt{p^2 + 1}, 1/2, x \right). \quad (5.7)$$

It is now checked that if $K_1(x)$ and $K_2(x)$ are substituted, Eq.(4.3) produces two independent functions, $M_1(x)$ and $M_2(x)$, respectively, which satisfy Eq.(4.7) for $M(x)$. Since, due to the hypergeometric differential equation, the second derivative of a hypergeometric function is expressed in terms of this function and its first derivative, it is readily understood by inspection of the structure of Eq.(4.3) that each of the functions $M_1(x)$ and $M_2(x)$ is written as a linear combination of two hypergeometric functions with rational coefficients.

In turn, Eq. (4.7) for $M(x)$ is factorized as

$$\left[\frac{d^2}{dx^2} + \frac{1}{2} \left(\frac{3}{x} - \frac{7}{1-x} \right) \frac{d(x)}{dx} + \frac{1}{4} \left(\frac{p^2 - a^2 - 9}{x} + \frac{p^2 - a^2 - 9}{1-x} - \frac{a^2 - 6}{(1-x)^2} \right) \right] g(x) = 0, \quad (5.8)$$

where

$$g(x) = \left[\frac{d^2}{dx^2} + \frac{1}{2} \left(\frac{1}{x} - \frac{3}{1-x} \right) \frac{d}{dx} + \frac{1}{4} \left(\frac{p^2 - a^2 - 1}{x} + \frac{p^2 - a^2 - 1}{1-x} - \frac{a^2}{(1-x)^2} \right) \right] M(x). \quad (5.9)$$

The equation $g(x) = 0$ is also solved in terms of the hypergeometric functions. The proper substitution is $M(x) = x^C(1-x)^D F(x)$:

$$C = 0, \quad \frac{1}{2}, \quad D = -\frac{1}{4} \pm \frac{1}{4} \sqrt{4a^2 + 1}, \quad F(x) = F(\alpha', \beta'; \gamma'; x), \quad (5.10)$$

$$\alpha' = C + D + \frac{1}{2} - \frac{p}{2}, \quad \beta' = C + D + \frac{1}{2} + \frac{p}{2}, \quad \gamma' = \frac{1}{2} + 2C. \quad (5.11)$$

Here again there are two choices for the parameters that lead to finite solutions:

$$(iii) \quad C = 1/2, \quad D = +j/2, \\ M_3 = x^{1/2} (1-x)^{j/2} F\left(1 + j/2 - \frac{p}{2}, 1 + j/2 + \frac{p}{2}, 3/2, x\right), \quad (5.12)$$

$$(iv) \quad C = 0, \quad D = +j/2, \\ M_4 = (1-x)^{j/2} F\left(j/2 + 1/2 - \frac{p}{2}, j/2 + 1/2 + \frac{p}{2}, 1/2, x\right). \quad (5.13)$$

In the similar way as was done above, it is checked that this time if $M_3(x)$ and $M_4(x)$ are substituted, Eq.(4.4) produces two independent functions, respectively, $K_3(x)$ and $K_4(x)$, which satisfy Eq.(4.6) for $K(x)$. It is again understood from Eq.(4.4) that each of the functions $K_3(x)$ and $K_4(x)$ is written as a linear combination of two hypergeometric functions with rational coefficients.

It is readily verified, e.g., by checking the Wronskian, that the functions K_1, K_2, K_3, K_4 are independent, hence, compose a set of fundamental solutions of Eq.(4.6) thus producing the general solution of this equation. Similarly, the functions M_1, M_2, M_3, M_4 produce the general solution of Eq.(4.7).

Passing from p to a quantity n via substitution $p^2(n)$ given as shown below, these functions are written as

$$(i) \quad p_{(1)}^2 = (j + 2 + 2n)^2 - 1, \quad n = 0, 1, 2, \dots, \\ K_1(x) = \sqrt{x} (1-x)^{j/2} F(-n, j + 2 + n; 3/2; x), \quad M_1(x), \quad (5.14)$$

$$(ii) \quad p_{(2)}^2 = (j + 1 + 2n)^2 - 1, \quad n = 0, 1, 2, \dots, \\ K_2(x) = (1-x)^{j/2} F(-n, j + 1 + n; 1/2; x), \quad M_2(x), \quad (5.15)$$

$$(iii) \quad p_{(3)}^2 = (j + 2n)^2, \quad n = 0, 1, 2, \dots, \\ K_3(x), \quad M_3(x) = \sqrt{x} (1-x)^{j/2} F(-n, j + 2 + n; 3/2; x), \quad (5.16)$$

$$(iv) \quad p_{(4)}^2 = (j+1+2n)^2, \quad n = 0, 1, 2, \dots, \\ K_4(x), \quad M_4(x) = (1-x)^{j/2} F(-n, j+1+n; 1/2; x), \quad (5.17)$$

where the lacking functions $M_1(x)$, $M_2(x)$ and $K_3(x)$, $K_4(x)$ are found from Eqs. (4.3) and (4.4):

$$(i) \quad M_1(x) = \frac{(1-x)^{j/2}}{\sqrt{j(j+1)}} \left(2n(x-1) {}_2F_1 \left(1-n, j+n+2; \frac{3}{2}; x \right) - \right. \\ \left. (jx+2n(x-1)+x-1) {}_2F_1 \left(-n, j+n+2; \frac{3}{2}; x \right) \right), \quad (5.18)$$

$$(ii) \quad M_2(x) = \frac{(1-x)^{j/2}}{\sqrt{j(j+1)}x} \left(2n(x-1) {}_2F_1 \left(1-n, j+n+1; \frac{1}{2}; x \right) - \right. \\ \left. (jx+2n(x-1)) {}_2F_1 \left(-n, j+n+1; \frac{1}{2}; x \right) \right), \quad (5.19)$$

$$(iii) \quad K_3(x) = \frac{(1-x)^{j/2}}{\sqrt{j(j+1)}} \left(2n(x-1) {}_2F_1 \left(1-n, j+n+2; \frac{3}{2}; x \right) - \right. \\ \left. ((j+2)x+2n(x-1)-1) {}_2F_1 \left(-n, j+n+2; \frac{3}{2}; x \right) \right), \quad (5.20)$$

$$(iv) \quad K_4(x) = \frac{(1-x)^{j/2}}{\sqrt{j(j+1)}\sqrt{x}} \left(2n(x-1) {}_2F_1 \left(1-n, j+n+1; \frac{1}{2}; x \right) - \right. \\ \left. ((j+1)x+2n(x-1)) {}_2F_1 \left(-n, j+n+1; \frac{1}{2}; x \right) \right). \quad (5.21)$$

For arbitrary a and p (i.e. if the parameters j and n are mathematically considered as arbitrary complex numbers), Eqs.(5.14)-(5.17) define the general solution of the fourth-order differential equations (4.6) and (4.7).

Since the derived general solution involves combinations of hypergeometric functions with rational coefficients, the discrete energy spectrum can be obtained by terminating the involved hypergeometric series. Indeed, this procedure necessarily results in quasi-polynomial solutions, which warrant discrete energy levels owing to the above-chosen particular asymptotes for the corresponding pre-factors for functions K_i, M_i . The condition for termination of a hypergeometric series is that an upper parameter of it is a negative integer. This is achieved if $n = 0, 1, 2, \dots$. Correspondingly, Eqs.(5.14)–(5.17) then define four sets of quasi-polynomial wave functions describing the discrete energy states of a Dirac-Kähler particle. The derived solutions reveal significant differences as compared with the result for the ordinary Dirac particle in spherical space.

The explicit form of the Dirac equation in the space model discussed here is

$$\left(i\gamma^0 \frac{\partial}{\partial t} + i\gamma^3 \frac{\partial}{\partial r} + \frac{1}{\sin r} \Sigma_{\theta\phi} - m \right) \Psi = 0 \quad (5.22)$$

and the general structure of the wave function corresponding to the diagonalization of operators $i\partial_t, \vec{J}^2, J_3$ is

$$\tilde{\Psi} = e^{-i\epsilon t} \begin{vmatrix} f_1(r) D_{-1/2} \\ f_2(r) D_{+1/2} \\ f_3(r) D_{-1/2} \\ f_4(r) D_{+1/2} \end{vmatrix}, \quad (5.23)$$

where the Wigner functions [35] are denoted as $D_\sigma = D_{-m,\sigma}^J(\phi, \theta, 0)$; J adopts half-integer values: $J = 1/2, 3/2, \dots$.

Omitting the details (see the discussion in [34]), the result for the energy spectrum of the Dirac particle is given as

$$p^2 = \epsilon^2 - m^2 = (n + J + 1)^2, \quad (5.24)$$

or in usual units

$$E^2 - m^2 c^4 = \frac{\hbar^2 c^2}{\rho^2} (n + J + 1)^2, \quad (5.25)$$

where ρ stands for the curvature radius.

Examining the derived spectrum (5.14)-(5.17) for the Dirac-Kähler particle, we note that all the four energy series differ from the energy spectrum (5.24) for the Dirac particle. Further, we note that $p_{(2)}^2 \rightarrow p_{(1)}^2$ and $p_{(4)}^2 \rightarrow p_{(3)}^2$ if j is shifted by unity: $j \rightarrow j + 1$. However, since the corresponding wave functions remain distinct, and thus present different states, we see that the energy spectrum of a Dirac-Kähler particle in a Riemann spherical space consists of two paralleled series each of which is twofold degenerate.

6 Discussion

Thus, we have shown that the energy spectrum of the Dirac-Kähler particle in spherical space is much more complicated compared with the spectrum for the Dirac particle. The difference is of both structural and quantitative character. The energy spectrum of a Dirac-Kähler particle consists of two paralleled series each of which is twofold degenerate. Besides, none of the separate energy level series coincides with that one for the Dirac particle.

We conclude with a brief discussion of some additional points concerning the relationship between the Dirac-Kähler field and a set of four Dirac fields in the case of flat Minkowski space. In order to explicitly connect the boson solutions of the Dirac-Kähler field with spherical solutions of the (four) Dirac equations, one should perform a special transformation $U(x) \rightarrow V(x)$ so chosen that in the new representation the Dirac-Kähler equation is splitted into four separated Dirac-type equations (see [33] for more details). Then, it is possible to decompose the four columns of the (4×4) -matrix $V(x)$, related with the Dirac-Kähler equation, in terms of the solutions of four Dirac-type equations. The mentioned transformation has the form

$$V(x) = (I \otimes S(x)) U(x), \quad S(x) = \begin{vmatrix} U(x) & 0 \\ 0 & U(x) \end{vmatrix},$$

$$U(x) = \begin{vmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} & \sin \frac{\theta}{2} e^{-i\phi/2} \\ -\sin \frac{\theta}{2} e^{+i\phi/2} & \cos \frac{\theta}{2} e^{+i\phi/2} \end{vmatrix}. \quad (6.1)$$

The spherical bispinor connection Γ_α :

$$\Gamma_t = 0, \quad \Gamma_r = 0, \quad \Gamma_\theta = j^{12}, \quad \Gamma_\phi = \sin \theta j^{32} + \cos \theta j^{12},$$

involved in the Dirac-Kähler connection $B_\alpha(x) = \Gamma_\alpha(x) \otimes I + I \otimes \Gamma_\alpha(x)$ (see Eq.(1.2)) in the $V(x)$ -representation is written as

$$\{ [i\gamma^\alpha(x) (\partial_\alpha + \Gamma_\alpha(x)) - m] V(x) + \\ + i\gamma^\alpha(x) V(x) [S(x) \Gamma_\alpha(x) S^{-1}(x) + S(x) \partial_\alpha S^{-1}(x)] \} = 0.$$

The chosen form (6.1) of $S(x)$ leads to vanishing the term

$$S(x) \Gamma_\alpha(x) S^{-1}(x) + S(x) \partial_\alpha S^{-1}(x) = 0 .$$

With this, we finally arrive at four disconnected Dirac equations for the columns of the matrix $V(x)$:

$$[i\gamma^\alpha(x) (\partial_\alpha + \Gamma_\alpha(x)) - m] V(x) = 0 . \quad (6.2)$$

Considering now the above spherical Dirac–Kähler solution, we thus should transform the solution from the U -form to a corresponding V -form, and second, should expand the four columns of the matrix V in terms of the Dirac spherical waves. The calculations performed in [33] lead to simple linear expansions of the four columns of the new representative of the Dirac–Kähler field $V(x)$ in terms of spherical fermion solutions $\Psi_i(x)$ of the four ordinary Dirac equations. It should be emphasized, however, that this procedure for construction of such expansions is applicable only for the case of flat Minkowski space-time; it cannot be extended to any space-time model with curvature.

Moreover, even in the case when such expansions exist, this fact cannot be interpreted as the equivalence of the Dirac–Kähler field and a system of four Dirac fermions. An essential objection against such an equivalence is that the mentioned transformation $(S(x) \otimes I)$ does not belong to the group of tetrad local gauge transformations for Dirac–Kähler field, which is a 2-rank bispinor under the Lorentz group. Therefore, the linear expansions between boson and fermion functions are not gauge invariant under the group of local tetrad rotations.

Finally, it should be also emphasized that the Wigner D -functions, related to integer and half-integer values of $j(J)$, involved, respectively, in the Dirac–Kähler wave functions $U(x)$ and the Dirac functions $\Psi(x)$, exhibit completely different boundary properties in angular variables (θ, ϕ) (see the details in [35]).

7 Acknowledgements

This work was supported by the Fund for Basic Researches of Belarus (Grant No. F14ARM-021) and by the Armenian State Committee of Science (Grant No. 13RB-052), within the cooperation framework between Belarus and Armenia. The research has been partially conducted within the scope of the International Associated Laboratory (CNRS-France & SCS-Armenia) IRMAS. The work by A. Ishkhanyan has received funding from the European Union Seventh Framework Programme, grant No. 295025 - IPERA).

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